On sequences of positive integers containing arithmetical progressions

T. Šalát* and J. Tomanová†

Abstract. *We study from the metrical and topological point of view the properties of sequences of positive integers which consist in fact that the sequences contain arbitrarily long arithmetical progressions and infinite arithmetical progressions, respectively. At the end of the paper we give another solution of the problem of R. C. Buck concerning the class $\mathcal{D}_\mu$ of all $A \subseteq N$ having Buck's measure $\mu(A)$.*

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1. Introduction

Denote by $\mathcal{U}$ the class of all infinite sequences of positive integers (in an increasing order). If $A \in \mathcal{U}$, $A = a_1 < a_2 < \cdots < a_n < \cdots$, then in agreement with [2] we put $A(n) = \sum_{a \leq n, a \in A} 1$, $\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}$ (the lower asymptotic density of A), $\overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$ (the upper asymptotic density of A) and if there exists $\lim_{n\to\infty} \frac{A(n)}{n}$, then we put $d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$ (the asymptotic density of A).

Let us remark that the symbol A will also denote the set of all terms of the sequence A.

We now introduce two fundamental results on arithmetical progressions in sequences $A \in \mathcal{U}$, which encouraged the preparation of this paper.

The first of these results is due to E. Szemerédi (cf. [3], [11]).

Theorem A. *If $\overline{d}(A) > 0$, then the sequence A contains arbitrarily long arithmetic progressions.*

The second result is due to S. S. Wagstaff, Jr. (cf [12]).

*Department of Algebra and Number Theory, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, SK - 84 215 Bratislava, Slovakia

†Department of Algebra and Number Theory, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, SK - 84 215 Bratislava, Slovakia, e-mail: Jana.Tomanova@fmph.uniba.sk

Theorem B. *To each* $a, b \in [0,1], a \leq b$ *there exists an* $A \in \mathcal{U}$ *such that* A *contains no infinite arithmetic progression and* $\underline{d}(A) = a, \overline{d}(A) = b$.

Denote by $\mathcal{F}$ ($\mathcal{I}$) the class of all such $A \in \mathcal{U}$ which contain arbitrarily long finite arithmetical progressions (which contain infinite arithmetical progressions). Evidently we have

$$\mathcal{I} \subseteq \mathcal{F}. \tag{1}$$

The study of systems $\mathcal{F}, \mathcal{I}$ from the metrical and topological point of view is enabled by the mapping $\rho : \mathcal{U} \to (0,1]$ defined in the following way: we put $\rho(A) = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, where $\varepsilon_k = 1$ if $k \in A$ and $\varepsilon_k = 0$ in the opposite case. It is easy to verify that ρ is a one-to-one mapping of $\mathcal{U}$ onto the interval $(0,1]$ (cf. [5], p. 17-18).

If $\mathcal{S} \subseteq \mathcal{U}$, then $\rho(\mathcal{S})$ denote the set of all numbers $\rho(A)$, where $A \in \mathcal{S}$. The study of properties of the set $\rho(\mathcal{S})$ gives us an image of the class $\mathcal{S}$ structure.

The purpose of this paper is the investigation of topological and metrical properties of the sets $\rho(\mathcal{F})$, $\rho(\mathcal{I})$ and $\rho(\mathcal{F} \setminus \mathcal{I})$.

In what follows $\lambda(M)$ and $\dim(M)$ denotes the Lebesgue measure and Hausdorff dimension of the set M, respectively. Further if $\mathcal{S} \subseteq \mathcal{U}$, then we put $\mathcal{S}^c = \mathcal{U} \setminus \mathcal{S}$.

2. Topological properties of sets $\rho(\mathcal{F}), \rho(\mathcal{I})$ and $\rho(\mathcal{F} \setminus \mathcal{I})$

In this part of the paper the interval $(0,1]$ is considered to be a metric space with the Euclidean metric.

Theorem 1. *The set* $\rho(\mathcal{F})$ *is residual in* $(0,1]$.

Proof. Since ρ is a one-to-one mapping, we have

$$\rho(\mathcal{F}) = (0,1] \setminus \rho(\mathcal{F}^c). \tag{2}$$

It suffices to prove that $\rho(\mathcal{F}^c)$ is a set of the first Baire category in $(0,1]$.

Denote by $\Gamma(0)$ the class of all such $A \in \mathcal{U}$ for which $d(A) = 0$. It follows from *Theorem A* that $\Gamma(0)^c \subseteq \mathcal{F}$. Hence $\mathcal{F}^c \subseteq \Gamma(0)$ and so we get

$$\rho(\mathcal{F}^c) \subseteq \rho(\Gamma(0)). \tag{3}$$

Express the numbers $x \in (0,1]$ by their non-terminating dyadic developments, hence $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$, $\varepsilon_k(x) = 0$ or $1 (k = 1, 2, \dots)$ and for an infinite number of k's we have $\varepsilon_k(x) = 1$. Put $N_n(1,x) = \sum_{k=1}^{n} \varepsilon_k(x)$. It follows from the main result of paper [8] that the set of all such $x \in (0,1]$ for which the limit points of the sequence $\left(\frac{N_n(1,x)}{n}\right)_1^{\infty}$ fill up the interval $[0,1]$, is residual in $(0,1]$. From this it follows at once that the set $\rho(\Gamma(0))$ is a set of the first category in $(0,1]$ and the assertion follows from (3). □

We can conclude from (1) that $\rho(\mathcal{I}) \subseteq \rho(\mathcal{F})$. This inclusion gives us no information about the topological size of the set $\rho(\mathcal{I})$. The following theorem together with *Theorem 1* gives us a qualitative image of the difference between the size of the classes $\mathcal{F}$ and $\mathcal{I}$.

On account of *Theorem B* it can be conjectured that class $\mathcal{I}$ is poor. The following theorem certificies this conjecture.

Theorem 2. *Set $\rho(\mathcal{I})$ is a set of the first Baire category in* $(0,1]$.

Proof. If $A \in \mathcal{U}$, $A = a_1 < a_2 < \cdots < a_k < a_{k+1} < \cdots$, then the numbers $a_{k+1} - (a_k + 1)$ $(k = 1, 2, \dots)$ are called the gaps of the sequence A. Denote by $\mathcal{V}$ the class of all such $A \in \mathcal{U}$ that the sequence of gaps of A is unbounded. Then we have evidently $\mathcal{V} \subset \mathcal{I}^c$ (cf. [12]) and therefore

$$\rho(\mathcal{I}) \subseteq \rho(\mathcal{V}^c). \tag{4}$$

Hence according to (4), it suffices to prove that $\rho(\mathcal{V}^c)$ is a set of the first category in $(0,1]$.

We have

$$\rho(\mathcal{V}^c) = \bigcup_{m=0}^{\infty} \rho(\mathcal{B}_m), \tag{5}$$

where $\mathcal{B}_m$ $(m = 0, 1, \dots)$ is the class of all such $A \in \mathcal{U}$ that the set of gaps of A is bounded from above by the number m.

It suffices to prove that each $\mathcal{B}_m$ $(m = 0, 1, \dots)$ is a nowhere dense set in $(0,1]$.

Let $m \geq 0$. The proof of the nowhere density of $\mathcal{B}_m$ will be realized by proving the following assertion (cf. [4], p. 37):

If $I \subset (0,1]$ is an arbitrary interval, then there exists such an interval $I^{'} \subset I$ that

$$I^{'} \cap \mathcal{B}_m = \emptyset. \tag{6}$$

Let I be an interval, $I \subset (0,1]$. Let us choose such integers $n \geq 1, 0 \leq j \leq 2^n - 1$ that

$$J = \left(j2^{-n}, (j+1)2^{-n}\right) \subset I. \tag{7}$$

It follows from the construction of dyadic expansions that there exists a finite sequence

$$a_1, a_2, \dots, a_n \tag{8}$$

of 0's and 1's such that for each $x \in J$, $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k}$ we have $\varepsilon_k(x) = a_k$ $(k = 1, 2, \dots n)$. For brevity we say that the interval J is associated with the sequence (8).

Let us construct the finite sequence

$$a_1, a_2, \dots, a_n, \underbrace{0, 0, \dots, 0}_{m+1 \text{ zeros}}.$$

Then all $x = \sum_{k=1}^{\infty} \varepsilon_k(x) 2^{-k} \in (0,1]$ with $\varepsilon_k(x) = a_k$ $(k = 1, 2, \dots, n)$ and $\varepsilon_k(x) = 0$ $(k = n+1, \dots, n+m+1)$ form an interval

$$I^{'} = \left(s2^{-n-m-1}, (s+1)2^{-n-m-1}\right), \quad I^{'} \subseteq I.$$

But $I^{'}$ fulfils (6) evidently. The proof is complete. □

In connection with classes $\mathcal{F}, \mathcal{I}$ it is convenient to investigate the class $\mathcal{F} \setminus \mathcal{I}$ of all $A \in \mathcal{U}$ that contain arbitrarily long finite arithmetical progressions but do not contain any infinite arithmetical progression.

Theorem 3. *The set $\rho(\mathcal{F} \setminus \mathcal{I})$ is residual in* $(0,1]$.

Proof. Since ρ is a one-to-one mapping, we have

$$\rho(\mathcal{F} \setminus \mathcal{I}) = \rho(\mathcal{F}) \cap \rho(\mathcal{I}^c). \tag{9}$$

According to *Theorem 1* and *Theorem 2*, each set on the right-hand side is residual in $(0,1]$. Hence the assertion follows at once from (9). □

We show that the sets $\rho(\mathcal{F})$, $\rho(\mathcal{I})$, $\rho(\mathcal{F} \setminus \mathcal{I})$ belong to the second Borel class in $(0,1]$.

Theorem 4.

(i) The set $\rho(\mathcal{F})$ is an $F_{\sigma\delta}-$set in $(0,1]$.

(ii) The set $\rho(\mathcal{I})$ is a $G_{\delta\sigma}-$set in $(0,1]$.

(iii) The set $\rho(\mathcal{F} \setminus \mathcal{I})$ is an $F_{\sigma\delta}-$set in $(0,1]$.

Proof. (i) For $a \geq 1$, $d \geq 1$, $m \geq 1$ denote by $\mathcal{F}(a,d,m)$ the class of all such $A \in \mathcal{U}$ that contain the finite sequence

$$a, a+d, \dots, a+md. \tag{10}$$

Put

$$\mathcal{F}(m) = \bigcup_{(a,d)} \mathcal{F}(a,d,m) \tag{11}$$

(the union of sets on the right-hand side is taken over all ordered pairs (a,d) of positive integers). We have

$$\mathcal{F} = \bigcap_{m=1}^{\infty} \mathcal{F}(m).$$

From this and from (11) we get

$$\rho(\mathcal{F}) = \bigcap_{m=1}^{\infty} \bigcup_{(a,d)} \rho(\mathcal{F}(a,d,m)). \tag{12}$$

It follows from the definition of the function ρ that for fixed a, d, m the set $\rho(\mathcal{F}(a,d,m))$ consists of the finite number of intervals of the form $(s2^{-j}, (s+1)2^{-j})$ $(j = a+dm, 0 \leq s \leq 2^j - 1)$ which are associated with sequences $\varepsilon_1, \varepsilon_2, \dots \varepsilon_j$ of 0's and 1's such that $\varepsilon_k = 1$ for $k = a, a+d, \dots, a+dm$ and $\varepsilon_k = 0$ or 1 for other k's, $k \leq j$.

It is obvious from the foregoing that $\rho(\mathcal{F}(a,d,m))$ is an F_σ-set in $(0,1]$ and the assertion follows from (12).

(ii) For fixed integers a, d, $a \geq 1$, $d \geq 1$ denote by $\mathcal{I}(a,d)$ the class of all such $A \in \mathcal{U}$ which contain the infinite arithmetical progression $a, a+d, \dots, a+kd, \dots$. Then we get

$$\mathcal{I} = \bigcup_{(a,d)} \mathcal{I}(a,d)$$

(the symbol $\cup_{(a,d)}$ has the same meaning as in (11)).

From this we have

$$\rho(\mathcal{I}) = \bigcup_{(a,d)} \rho(\mathcal{I}(a,d)). \tag{13}$$

Further, we set

$$H_m = \rho(\mathcal{F}(a,d,m)), m = 1,2,\dots$$

(the symbol $\mathcal{F}(a,d,m)$ has the same meaning as at the beginning of the proof).

From the definition of H_m, $m = 1,2,\dots$ we obtain

$$H_1 \supseteq H_2 \supseteq \dots \supseteq H_k \supseteq \dots .$$

So we get

$$\rho(\mathcal{I}(a,d)) = \bigcap_{m=1}^{\infty} H_m.$$

Since H_m is both an F_σ and G_δ set in $(0,1]$ $(m = 1,2,\dots)$, the set $\rho(\mathcal{I}(a,d))$ is a G_δ set in $(0,1]$, too and the assertion follows from (13).

(iii) Let us consider that

$$\rho(\mathcal{I}^c) = (0,1] \setminus \rho(\mathcal{I})$$

and use (9). The assertion follows. □

3. Metric properties of the sets $\rho(\mathcal{F})$, $\rho(\mathcal{I})$ and $\rho(\mathcal{F} \setminus \mathcal{I})$

We show that the class $\mathcal{F}$ is also rich from the metric point of view (compare the following theorem with *Theorem 1*).

Theorem 5. *We have* $\lambda(\rho(\mathcal{F})) = 1$.

Proof. Denote by $\mathcal{T}$ the class of all such $A \in \mathcal{U}$ for which $d(A) = \frac{1}{2}$. Then *Theorem A* implies that $\mathcal{T} \subseteq \mathcal{F}$, hence $\rho(\mathcal{T}) \subseteq \rho(\mathcal{F})$. But it is well-known that $\lambda(\rho(\mathcal{T})) = 1$ ([5], p. 190). The assertion follows. □

Theorem 6. *We have* $\lambda(\rho(\mathcal{I})) = 0$.

Proof. It follows from (13) that

$$\lambda(\rho(\mathcal{I})) \le \sum_{(a,d)} \lambda(\rho(\mathcal{I}(a,d))). \tag{14}$$

It is not difficult to show that $\lambda(\rho(\mathcal{I}(a,d))) = 0$ for each a, d (see also Lemma in [10]). The assertion follows from (14). □

Let us notice that *Theorems 5* and *6* give an analogous look on the size of the classes $\mathcal{F}, \mathcal{I}$, as *Theorems 1* and *2*.

From *Theorem 5* and *Theorem 6* we get

Theorem 7. *We have* $\lambda(\rho(\mathcal{F} \setminus \mathcal{I})) = 1$.

The problem appears to determine the precise value of the Hausdorff dimensions of the null-sets $\rho(\mathcal{F}^c)$, $\rho(\mathcal{I})$. The problem is solved in the following

Theorem 8. *We have*

(i) $\dim(\rho(\mathcal{F}^c)) = 0$.

(ii) $\dim(\rho(\mathcal{I})) = 1$.

Proof. (i) It is well-known that $\dim(\Gamma(0)) = 0$ (cf. [5], p. 195). But then from (3) we get $\dim(\rho(\mathcal{F}^c)) = 0$.

(ii) It follows from (13) that for each positive integer d we have

$$\rho(\mathcal{I}) \supset \rho(\mathcal{I}(1,d)). \tag{15}$$

We shall determine $\dim(\rho(\mathcal{I}(1,d)))$. Let us consider that $\rho(\mathcal{I}(1,d))$ is the set of all such $x = \sum_{k=1}^{\infty} \varepsilon_k(x)2^{-k} \in (0,1]$ for which $\varepsilon_{1+jd}(x) = 1$ $(j = 0,1,\dots)$ and $\varepsilon_k(x) = 0$ or 1 for $k \neq 1 + jd$ $(j = 0,1,\dots)$.

From Theorem 2.7 of paper [9] the following result can be easily deduced:

$$\dim(\rho(\mathcal{I}(1,d))) = \liminf_{n\to\infty} \frac{\log \prod\limits_{k\le n, k\neq 1+jd} 2}{n \log 2}. \tag{16}$$

By a simple calculation for the number p_n of k's, $k \le n, k \neq 1+jd$ $(j = 0,1,\dots)$ we get the equality

$$p_n = n - \left(\left[\frac{n-1}{d} \right] + 1 \right)$$

and this used in (16) yields

$$\dim(\rho(\mathcal{I}(1,d))) = 1 - \frac{1}{d}. \tag{17}$$

From (15) and (17) we obtain

$$\dim \rho(\mathcal{I}) \ge 1 - \frac{1}{d}.$$

This is true for each positive integer d, hence by $d \to +\infty$ we get $\dim(\rho(\mathcal{I})) = 1$. $\square$

At the end of our considerations we shall use the foregoing results to give another solution of a problem formulated by R. C. Buck. In [1] a finitely additive measure μ is defined on an algebra $\mathcal{D}_\mu$ of sets $A \subseteq \mathcal{U}$ which contains all sets that are finite unions of arithmetic progressions or which differ from these by finite sets. It is proved in [1] that $\mathcal{D}_\mu$ has the power of the continuum, and $\rho(\mathcal{D}_\mu)$ is a set of the first Baire categoty in $(0,1]$. The author says on p. 580: "It would be of interest to know if this set (i.e. $\rho(\mathcal{D}_\mu)$) is measurable, and if so, whether its measure is zero or one".

The answer to this question was already given by M. Parnes in [6]. We shall give a quite different solution, based on *Theorem 6* and the following result:

Theorem C (Theorem 3.1 in [7]). *If $A \in \mathcal{D}_\mu$ and $\mu(A) > 0$, then A belongs to $\mathcal{I}$.*

Theorem 9. *We have $\lambda(\rho(\mathcal{D}_\mu)) = 0$.*

Proof. Evidently we have $\mathcal{D}_\mu = \{A \in \mathcal{D}_\mu; \mu(A) > 0\} \bigcup \{A \in \mathcal{D}_\mu; \mu(A) = 0\}$. Hence according to *Theorem C*, we get

$$\rho(\mathcal{D}_\mu) \subseteq \rho(\mathcal{I}) \cup \rho(\mathcal{W}_0),$$

where $\mathcal{W}_0 = \{A \subseteq N; d(A) = 0\}$. But $\lambda(\rho(\mathcal{I})) = 0$ (*Theorem 6*) and $\lambda(\rho(\mathcal{W}_0)) = 0$ (This is an easy consequence of Borel's theorem on distribution of digits in dyadic developments of real numbers - see [5], p. 190.).

Hence the set $\rho(\mathcal{D}_\mu)$ as a subset of a set of Lebesgue measure zero is L-measurable and its measure equals zero. □

In connection with *Theorem 9* the question arises whether $\rho(\mathcal{D}_\mu)$ is a Borel set, and if so, which Borel clas it belongs to. Further it would be interesting to determine the Hausdorff dimension of $\rho(\mathcal{D}_\mu)$.

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